Row Echelon Form (REF)

A matrix is in row echelon form if and only if
the first (leftmost) non-zero entry in each row is 1 (called the leading 1),
the leading 1 in each row (except row 1) is to the right of the leading 1 in the row above it,
and all rows which contain only 0 are below all rows which contain any non-zero entry.

A matrix in REF corresponds to a system of equations that needs only back-substitution to solve.

Are these matrices in REF ? If not, why not ?

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 4 \\ 0 & 1 & 7 & 4 & 0 \\ 0 & 0 & -1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & -2 & 4 \\ 0 & 1 & 7 & 4 & 0 \\ 0 & 1 & 4 & -3 & -2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & -2 & 4 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & -2 & 4 \\ 0 & 1 & 7 & 4 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form (RREF)

A matrix is in reduced row echelon form if and only if
it is in row echelon form,
and all columns which contain a leading 1 contain only 0 in all other entries.

A matrix in RREF corresponds to a system of equations that needs the least amount of algebra to solve.

Are these matrices in RREF ? If not, why not ?

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 4 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination Pivot Method

Step 1: Find the first (leftmost) column which contains a non-zero entry
Step 2: Choose a pivot in that column (to be used to replace all lower entries in that column with 0)
Step 3: SWAP to move the pivot's row to the top
Step 4: SCALE to turn the pivot into 1
Step 5: REPLACE each row below the pivot's row
by adding the multiple of the pivot's row which gives a 0 under the pivot
Step 6: Cover up the pivot's row & repeat the entire process (stop when matrix is in row echelon form)

Gauss-Jordan Elimination (after matrix is in row echelon form)

Step 7: Find the last (rightmost) column which contains a pivot (leading 1)
Step 8: REPLACE each row above the pivot's row
by adding the multiple of the pivot's row which gives a 0 above the pivot
Step 9: Cover up the pivot's row & repeat the entire process (stop when matrix is in reduced row echelon form)

The following examples should not require fractions if solved using the processes above.

Example 1:

$3x + 2y - z = -1$
$5x + y - 3z = -2$
$2x + 4y + 2z = 2$

Example 2:

$4x + 6y - 3z = -5$
$3x + 4y + z = 11$
$-x - 2y + z = 1$

Example 3:

$3x + 4y - 11z = -17$
$2x + y - 4z = 5$
$-x - 2y + 5z = -9$

Example 4:

$3x + 5y - 9z = 14$
$2x - 3y + 13z = 3$
$-x + 2y - 8z = -1$

Example 5:

$2x + 4y + 11z = 10$
$x + 2y + 7z = 5$
$3x + 4y + 9z = 13$

Example 1:

CHOOSE 2 in column 1, row 3 as pivot
(to avoid fractions after scaling)

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -1 \\ 5 & 1 & -3 & -2 \\ 2 & 4 & 2 & 2 \end{array}\right] R_1 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 2 & 2 \\ 5 & 1 & -3 & -2 \\ 3 & 2 & -1 & -1 \end{array}\right] R_1 \times \tfrac{1}{2} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 5 & 1 & -3 & -2 \\ 3 & 2 & -1 & -1 \end{array}\right] \begin{array}{l} R_2 + (-5)R_1 \\ R_3 + (-3)R_1 \end{array} \Rightarrow$$

SWAP to move pivot to top row — SCALE to turn pivot into 1 — REPLACE to eliminate all entries below pivot

COVER row 1 until matrix in REF
CHOOSE -4 in column 2, row 3 as pivot
(to avoid fractions after scaling)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -9 & -8 & -7 \\ 0 & -4 & -4 & -4 \end{array}\right] R_2 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & -9 & -8 & -7 \end{array}\right] R_2 \times \tfrac{1}{-4} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -9 & -8 & -7 \end{array}\right] R_3 + (9)R_2 \Rightarrow$$

SWAP to move pivot to top row — SCALE to turn pivot into 1 — REPLACE to eliminate all entries below pivot

COVER row 2 until matrix in REF
CHOOSE 1 in column 3, row 3 as pivot

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \textbf{ REF} \Rightarrow$$

SWAP to move pivot to top row
UNNECESSARY

SCALE to turn pivot into 1
UNNECESSARY

Rightmost leading 1 in column 3 is pivot

COVER row 3 until matrix in RREF
Rightmost leading 1 in column 2 is pivot

COVER row 2 until matrix in RREF

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 + (-1)R_3 \\ R_2 + (-1)R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] R_1 + (-2)R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \textbf{ RREF} \Rightarrow \begin{array}{l} x = 1 \\ y = -1 \\ z = 2 \end{array}$$

REPLACE to eliminate all entries above pivot — REPLACE to eliminate all entries above pivot

CHECK:

$3(1) + 2(-1) - (2) = 3 - 2 - 2 = 1$
$5(1) + (-1) - 3(2) = 5 - 1 - 6 = -2$
$2(1) + 4(-1) + 2(2) = 2 - 4 + 4 = 2$